A definable strong G retract of a definable G set in a real closed field

Tomohiro Kawakami
Department of Mathematics, Faculty of Education, Wakayama University,
Sakaedani Wakayama 640-8510, Japan
kawa@center.wakayama-u.ac.jp

Abstract

Let G be a definably compact definable group and X a definable G set. We prove that there exists a definable strong G deformation retraction L from X to a definably compact definable G subset Y of X.

2010 *Mathematics Subject Classification*. 14P10, 57S10, 03C64.
Keywords and Phrases. O-minimal, real closed fields, definable G sets, definably compact, definable G CW complexes.

1 . Introduction.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$, every definable map is assumed to be continuous and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$" unless otherwise stated.

General references on o-minimal structures are [2], [3], also see [8].

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the field $\mathbb{R}$ of real numbers. Definable C^rG manifolds and definable G sets in $\mathcal{M}$ are studied in [6], [5], [4].

In $\mathcal{M}$ the following theorem is proved ([5]).

Theorem 1.1 ([5]). *Let G be a compact definable group and X a definable G set. Then there exists a definable strong G deformation retraction L from X to a compact definable G subset Y of X.*

In this paper, we generalize Theorem 1.1 to $\mathcal{N}$.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

A definable subset X of R^n is *definably compact* if for every definable map $f : [0,1)_R \to X$, there exists the limit $\lim_{x \to 1} f(x)$ in X, where $[0,1)_R = \{x \in R | 0 \le x < 1\}$. If $R = \mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is a *representation* if it is definable, where $O_n(R)$ means the nth orthogonal group of R. A *representation space* of G is R^n with the orthogonal action induced from a representation of G. A *definable G set* means a G invariant definable subset of some representation space of G.

Theorem 1.2. *Let G be a definably compact definable group and X a definable G set. Then there exists a definable strong G deformation retraction L from X to a definably compact definable G subset Y of X.*

2 Proof of Theorem 1.2

Let $X \subset R^n, Z \subset R^m$ be definable sets and $f : X \to Z$ a definable map. We say that f is a *definable homeomorphism* if there exists a definable map $h : Z \to X$ such that $f \circ h = id_Z$ and $h \circ f = id_X$. We call f *definably proper* if for every definably compact subset C of Z, $f^{-1}(C)$ is definably compact.

Theorem 2.1. *(1) (Definable triangulation (e.g. (8.2.9 [2])). Let $S \subset R^n$ be a definable set and $S_1, \dots, S_k$ definable subsets of S. Then there exist a finite simplicial complex K in R^n and a definable map $\phi : S \to R^n$ such that ϕ maps S and each S_i definably homeomorphically onto a union of open simplexes of K. If S is definably compact, then we can take $K = \phi(S)$.*
(2) (Piecewise definable trivialization (e.g. 9.1.2 [2])). Let X and Z be definable sets and $f : X \to Z$ a definable map. Then there exist a finite partition $\{T_i\}_{i=1}^k$ of Z into definable sets and definable homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(z_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \le i \le k)$, where $z_i \in T_i$ and $p_i : T_i \times f^{-1}(z_i) \to T_i$ denotes the projection.
(3) (Existence of definable quotient (e.g. 10. 2.18 [2])). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.

A subgroup of a definable group is a *definable subgroup* of it if it is a definable subset of it. Note that every definable subgroup of a definable group is closed ([7]) and a closed subgroup of a definable group is not necessarily definable. A definable map (resp. A definable homeomorphism) between definable G sets is a *definable G map* (resp. a *definable G homeomorphism*) if it is a G map.

Let G be a definable group. A *definable set with a definable G action* is a pair (X, ϕ) consisting of a definable set X and a group action $\phi : G \times X \to X$ such that ϕ is a definable map. This action is not necessarily linear (orthogonal). Similarly, we can define *definable G maps* and *definable G homeomorphisms* between them.

Using Theorem 2.1 (3), if H is a definable subgroup of a definably compact definable group G, then G/H is a definable set, and the standard action $G \times G/H \to G/H$ defined by $(g, g'H) \mapsto gg'H$ of G on G/H makes G/H a definable set with a definable G action.

Definition 2.2. Let G be a definably compact definable group.

(1) A *definable G CW complex* is a finite G CW complex $(X, \{c_i | i \in I\})$ satisfying the following three conditions.

(a) The underlying space $|X|$ of X is a definable G set.

(b) The characteristic map $f_{c_i} : G/H_{c_i} \times \Delta \to \overline{c_i}$ of each open G cell c_i is a definable G map and $f_{c_i}|G/H_{c_i} \times \text{Int}\, \Delta : G/H_{c_i} \times \text{Int}\, \Delta \to c_i$ is a definable G homeomorphism, where H_{c_i} is a definable subgroup of G, Δ denotes a standard closed simplex, $\overline{c_i}$ is the closure of c_i in X, and Int Δ means the interior of Δ.

(c) For each c_i, $\overline{c_i} - c_i$ is a finite union of open G cells.

(2) Let X and Z be definable G CW complexes. A cellular G map $f : X \to Z$ is *definable* if $f : |X| \to |Z|$ is definable.

Since G and every standard closed simplex are definably compact and by definition, every definable G CW complex X is definably compact. Note that a G CW subcomplex of a definable G CW complex is a definable G CW complex itself.

Theorem 2.3. *Let G be a definably compact definable group. Let X be a definable G set and Y a definable closed G subset of X. Then there exist a definable G CW complex Z in a representation space Ξ of G, a G CW subcomplex W of Z, and a definable G map $f : X \to Z$ such that:*

1. *f maps X and Y definably G homeomorphically onto G invariant definable subsets Z_1 and W_1 of Z and W obtained by removing some open G cells from Z and W, respectively.*
2. *The orbit map $\pi : Z \to Z/G$ is a definable cellular map.*
3. *The orbit space Z/G is a finite simplicial complex compatible with $\pi(Z_1)$ and $\pi(W_1)$.*
4. *For each open G cell c of Z, $\pi|\overline{c} : \overline{c} \to \pi(\overline{c})$ has a definable section $s : \pi(\overline{c}) \to \overline{c}$, where $\overline{c}$ denotes the closure of c in Z.*

Moreover, if X is definably compact, then $Z = f(X)$ and $W = f(Y)$.

By a way similar to Theorem II.3.1 [1] and 2.7 [6], we have the following lemma.

Lemma 2.4. *Let G be a definably compact definable group, K, H definable subgroups of G with $K < H$ and X is a definable K set. Then the map $G \times_K X \to G \times_H (H \times_K X), [g, x] \mapsto [g, [e, x]]$ is a definable G homeomorphism, where e denotes the unit element of G.*

As in a similar way of [4], we can define orbit types of a definably compact definable group and we have the following theorem.

Theorem 2.5. *Let G be a definably compact definable group. Then every definable G set has only finitely many orbit types.*

Using Theorem 2.1 (2), (3), Lemma 2.4 and Theorem 2.5, by a way similar to the proof of 2.5 [6], we have the following theorem. It is an equivariant version of Theorem 2.1 (2) and a generalization of 2.5 [6].

Theorem 2.6. *Let G be a definably compact definable group, X a definable G set, Z a definable set and $f : X \to Z$ a G invariant definable map. Then there exist a finite decomposition $\{T_i\}_{i=1}^k$ of Z into definable sets and definable G homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(z_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \le i \le k)$, where p_i denotes the projection $T_i \times f^{-1}(y_i) \to T_i$ and $z_i \in T_i$.*

Proof of Theorem 2.3. Let Ω be a representation space of G containing X as a G invariant definable subset. We identify Ω with $\Omega \times \{1\} \subset \Omega \times R := \Xi$. Replacing Ω by Ξ, we may assume that $0 \notin X$. Let $\psi : \Xi - \{0\} \to \Xi - \{0\}$ be the definable map defined by $x \mapsto x/||x||^2$, where $||x||$ denotes the standard norm of x. By the definition of definable G sets, the standard norm is G invariant. Thus ψ is a definable G homeomorphism. Replacing X by $\psi(X)$, we may assume that X is bounded. Then the closure $\overline{X}$ of X in Ξ is a definably compact definable G set. By Theorem 2.1 (3), $\overline{X}/G$ is a definably compact definable set and the orbit map $\pi_{\overline{X}} : \overline{X} \to \overline{X}/G$ is a definable map.

By Theorem 2.6, there exist a finite decomposition $\{B_i\}_{i=1}^k$ of $\overline{X}/G$ into definable sets and definable G homeomorphisms $\phi_i : B_i \times \pi_{\overline{X}}^{-1}(b_i) \to \pi_{\overline{X}}^{-1}(B_i)$, $(1 \le i \le k)$, such that $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(B_i) = p_i \circ \phi_i^{-1}$, $(1 \le i \le k)$, where $b_i \in B_i$ and p_i denotes the projection $B_i \times \pi_{\overline{X}}^{-1}(b_i) \to B_i$.

By Theorem 2.1 (1) and since $\overline{X}/G$ is definably compact, there exist a finite simplicial complex $\overline{K}$ and a definable homeomorphism $\tau : \overline{X}/G \to \overline{K}$ such that τ maps each of $\pi_{\overline{X}}(X), \{B_i\}, \pi_{\overline{X}}(Y), cl(\pi_{\overline{X}}(Y))$ onto a union of open simplexes of K, where $cl(\pi_{\overline{X}}(Y))$ denotes the closure of $\pi_{\overline{X}}(Y)$ in $\overline{X}/G$. Note that $\tau(cl(\pi_{\overline{X}}(Y)))$ is a subcomplex of $\overline{K}$. Replace $\overline{K}$ by its first barycentric subdivision K.

We claim that each closed simplex $\Delta \in K$ admits a definable section $s : \tau^{-1}(\Delta) \to \pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$ of $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$.

By the choice of K, for every open simplex Int Δ, there exists a definable G homeomorphism $h : \pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta)) \to \pi_{\overline{X}}^{-1}(a) \times \tau^{-1}(\text{Int } \Delta)$ such that $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta)) = p' \circ h$, where $p' : \pi^{-1}(a) \times \tau^{-1}(\text{Int } \Delta) \to \tau^{-1}(\text{Int } \Delta)$ denotes the projection onto the second factor and $a \in \tau^{-1}(\text{Int } \Delta)$. Hence we obtain a definable section $\tilde{s}$ of $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta))$ defined by $\tilde{s}(x) = h^{-1}(b, x)$, where $b \in \pi_{\overline{X}}^{-1}(a)$. Since $\overline{X}$ is definably compact, Δ is a closed simplex and h is definable, we have a definable extension $s : \tau^{-1}(\Delta) \to \pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$ of $\tilde{s}$. Thus the proof of the claim is complete.

Put $\sigma = s(\tau^{-1}(\Delta))$. Then $s \circ \tau^{-1} : \Delta \to \sigma$ is a definable homeomorphism. Hence there exists a definable G map $f_\sigma : G/H \times \Delta \cong G(b) \times \Delta \to G\sigma, (gH, x) \mapsto g(s\tau^{-1}(x)))$ such that $f_\sigma|G/H \times \text{Int } \Delta : G/H \times \text{Int } \Delta \to G\sigma$ is a definable G homeomorphism, where H denotes the isotropy subgroup of b. Moreover f_σ is a definable G homeomorphism.

By collecting G cells $G\sigma = \pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$ for all closed simplexes Δ of K, we have a definable G CW complex Z such that $Z = \overline{X}$ and $Z/G = \overline{X}/G$. Similarly we obtaine a subcomplex W of Z such that $W = \overline{Y}$ and $W/G = \overline{Y}/G$, where $\overline{Y}$ denotes the closure of Y in Ξ. By the construction of Z, the orbit map $\pi : Z \to Z/G$ is a definable cellular map. Taking $Z_1 = \cup\{\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta))|\Delta \in K, \tau^{-1}(\text{Int } \Delta) \subset \pi_{\overline{X}}(X)\}$ and $W_1 = \cup\{\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta))|\Delta \in K, \tau^{-1}(\text{Int } \Delta) \subset \pi_{\overline{X}}(Y)\}$, we have the required definable G homeomorphism f from (X, Y) to (Z_1, W_1). $\square$

Note that in the proof of Theorem 2.3, replacing K by any subdivision K^* of K, we have the corresponding subdivision of Z^* of Z instead of Z.

Let X be a definable G set and Y a definable G subset of X. We say that a definable G map $l : X \to Y$ is a *definable G retraction from X to Y* if $l|Y = id_Y$. A *definable strong G deformation retraction from X to Y* is a definable G map $L : X \times [0,1]_R \to X$ such that $L(x, 0) = x$ for all $x \in X$, $L(y, t) = y$ for all $y \in Y, t \in [0,1]_R$ and $L(X, 1) = Y$, where the action on $[0,1]_R = \{x \in R | 0 \le x \le 1\}$ is trivial. Note that $L(\cdot, 1) : X \to Y$ is a definable G retraction from X to Y.

Let Z be a finite simplicial complex in R^n and X a union of open simplexes of Z. A subset Y of X is called a *subcomplex* of X if there exists a subcomplex Z_1 of Z with $Y = X \cap Z_1$. Note that every subcomplex of X is closed in X. The *first barycentric subdivision X' of X* is the intersection of the first barycentric subdivision Z' of Z with X. Similarly the nth barycentric subdivision of X is defined. The *star $St_X(Y)$* (resp. $St_{X'}(Y)$) *of Y in X* (resp. X') is the union of all open simplexes σ of X (resp. X') with $cl(\sigma) \cap Y \neq \emptyset$, where $cl(\sigma)$ denotes the closure of σ in X.

The above terms are defined similarly for definable G CW complexes.

Let X be a union of open simplexes of a finite simplicial complex Z. Then the maximal definably compact subcomplex Y of X is $\{\sigma \in Z | \overline{\sigma} \subset X\}$ and $X = St_X(Y)$, where $\overline{\sigma}$ denotes the closure of σ in Z.

Proof of Theorem 1.2. Let Ξ be a representation space of G containing X as a definable G set. Then by Theorem 2.3, X is definably G homeomorphic to a union of open G cells of a definable G CW complex C in Ξ. We identify X with its definably G homeomorphic image and replace C and X by their second barycentric subdivisions.

Let $f_c : G/H \times \Delta \to \overline{c} \subset C$ be the definable characteristic map of an open G cell c of X and put $\sigma = f_c(\{eH\} \times \text{Int } \Delta)$, where $\overline{c}$ denotes the closure of c in C. Remark that $c = G\sigma$ and $\overline{c} = G\overline{\sigma} = \overline{G\sigma}$, where $\overline{\sigma}$ denotes the closure of σ in C.

Let Y denote the maximum definably compact G CW subcomplex of X. In other words, Y is the union of all open G cells c of X such that $\overline{c} \subset X$. Then $\overline{c} \cap Y \neq \emptyset$ for all open G cells c of X, thus the star $St_X(Y)$ of Y in X is X.

Let C_n be the set of open G n-cells c of X such that $c \cap Y = \emptyset$. Then each C_n is a finite set and $C_0 = \emptyset$. Let $X_0 = Y$ and $X_n = Y \cup X^{(n)}$ for $n \geq 1$, where $X^{(n)}$

denotes the union of open G r-cells c of X with $r \leq n$. Then $X_n = Y \cup \cup_{c \in \cup_{k=0}^{n} C_k} c$.

By the construction of a definable G CW complex structure C of X, for every open G n-cell $c \in C_n$, there exists a proper subset Δ' of Δ obtained by removing some lower dimensional faces of Δ such that $f_c^{-1}(\overline{c} \cap X) = G/H \times \Delta'$. By the construction of Y, if $\overline{c} \subset X$, then $\overline{c} \subset Y$ Let $\delta = f_c(\{eH\} \times \Delta')$. Then $\sigma \subset \delta \subsetneqq \overline{\sigma} = f_c(\{eH\} \times \Delta)$, cl $\sigma = \delta$ and $G\delta =$ cl c, where cl σ (resp. cl c) denotes the closure of σ (resp. c) in X.

Remark that there exists a semialgebraic strong deformation retraction $\Delta' \times [0,1]_R \to \Delta'$ from Δ' to $\partial \Delta' := \Delta' - \mathrm{Int}\,\Delta'$. Thus for every open G n-cell $c = G\sigma \in C_n$, there exists a definable strong H deformation retraction $F_\delta^n : \delta \times [0,1]_R \to \delta$ from δ to $\partial\delta := \delta - \mathrm{Int}\,\delta$, because the action H action on δ is trivial. Using F_δ^n, we have a definable strong G deformation retraction

$$L_{G\delta}^n := G \times_H F_\delta^n : (G \times_H \delta) \times [0,1]_R \to G \times_H \delta$$

from $G \times_H \delta$ to $G \times_H \partial\delta$. Since $G \times_H \delta \cong G\delta$ and $G \times_H \partial\delta \cong G\partial\delta$, it gives a definable strong G deformation retraction from $G\delta$ to $G\partial\delta$ ($\subset X_{n-1}$).

Thus $\cup\{L_{G\delta}^n | c \in C_n\}$ induces a definable strong G deformation retraction $L^n : X_n \times [0,1]_R \to X_n$ from X_n to X_{n-1}. We can define $L^{n-1} \bullet L^n : X_n \times [0,1]_R \to X_n$, $L^{n-1} \bullet L^n(x,t)$

$$= \begin{cases} L^n(x,2t), & 0 \leq t \leq 1/2 \\ L^{n-1}(L^n(x,1), 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Therefore the required definable strong G deformation retraction $L = L^1 \bullet L^2 \bullet \cdots \bullet L^{m-1} \bullet L^m : X \times [0,1]_R \to X$ from X to Y is obtained inductively, where $m = \min\{n \in \mathbb{N} | X = X_n\}$. $\square$

References

[1] G.E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, **46** Academic Press, New York-London, (1972).

[2] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).

[3] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.

[4] T. Kawakami, *Definable C^r groups and proper definable actions*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **58** (2008), 9–18.

[5] T. Kawakami, *Definable G CW complex structures of definable G sets and their applications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **54** (2004), 1–15.

[6] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323-349.

[7] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra **53** (1988), 239-255.

[8] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Mathematics **150**, Birkhäuser, Boston, 1997.